

Efficient Simulation of Quantum States Based on Classical Fields Modulated with Pseudorandom Phase Sequences

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(Dated: today)

Abstract

We demonstrate that a tensor product structure could be obtained by introducing pseudorandom phase sequences into classical fields with two orthogonal modes. Using classical fields modulated with pseudorandom phase sequences, we discuss efficient simulation of several typical quantum states, including product state, Bell states, GHZ state, and W state. By performing quadrature demodulation scheme, we could obtain the mode status matrix of the simulating classical fields, based on which we propose a sequence permutation mechanism to reconstruct the simulated quantum states. The research on classical simulation of quantum states is important, for it not only enables potential practical applications in quantum computation, but also provides useful insights into fundamental concepts of quantum mechanics.

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I. INTRODUCTION

The classical simulation of quantum systems, especially of quantum entanglement has been under investigation for a long time [1–3]. In addition to potential practical applications in quantum computation, research on classical simulation systems could help understand some fundamental concepts in quantum mechanics. However, it has been pointed out by several authors that classical simulation of quantum systems exhibit exponentially scaling of physical resources with the number of quantum particles [3, 4]. In Ref. [3], an optical analogy of quantum systems is introduced, in which the number of light beams and optical components required grows exponentially with the number of cebits. In Ref. [5], a classical protocol to efficiently simulate any pure-state quantum computation is presented, yet the amount of entanglement involved is restricted. In Ref. [4], it is elucidated that in classical theory, the state space of a composite system is the Cartesian product of subsystems, whereas in quantum theory it is the tensor product. This essential distinction between Cartesian and tensor products is precisely the phenomenon of quantum entanglement, and viewed as the origin of the limitation of classical simulation of quantum systems.

In wireless and optical communications, orthogonal pseudorandom sequences have been widely applied to Code Division Multiple Access (CDMA) communication technology as a way to distinguish different users [6, 7]. A set of pseudorandom sequences is generated by using a shift register guided by a Galois field $GF(p)$, that satisfies orthogonal, closure and balance properties [7]. In Phase Shift Keying (PSK) communication systems, pseudorandom sequences are used to modulate the phase of the electromagnetic/optical wave, where a pseudorandom sequence is mapped to a pseudorandom phase sequence (PPS) values in $\{0, 2\pi/p, \dots, 2\pi(p-1)/p\}$. Guaranteed by the orthogonal property of the PPS, different electromagnetic/optical waves could transmit in one communication channel simultaneously with no crosstalk, and could be easily distinguished by implementing a quadrature demodulation measurement [6].

In this paper, by introducing the PPSs into classical fields, we explore an efficient simulation of quantum states based on classical fields with two orthogonal modes. We demonstrate that n classical fields modulated with n different PPSs can constitute a 2^n -dimensional Hilbert space that contains tensor product structure, which is similar with quantum systems. In Ref. [8], the efficient classical simulation of Bell states and GHZ state has been

introduced and both correlation analysis and von Neumann entropy have been applied to characterize the simulation. In this paper, by performing quadrature demodulation scheme, we could obtain the mode status matrix of the simulating classical fields, based on which we propose a sequence permutation mechanism to reconstruct the simulated quantum states. Besides, classical simulation of some other typical quantum states is discussed, including product state and W state. We generalize our simulation and discuss the efficiency of our simulation in the final end.

The paper is organized as follows: In Section II, we introduce some preparing knowledge needed later in this paper. In Section III, the existence of the tensor product structure in our simulation is demonstrated and the classical simulation of several typical quantum states is analyzed. In Section IV, a generalization of our simulation is proposed and the efficiency of the simulation is discussed. Finally, we summarize our conclusions in Section V.

II. PREPARING KNOWLEDGE

In this section, we introduce some notation and basic results needed later in this paper. We first introduce pseudorandom sequences and their properties. Then we discuss the similarities between classical field and single-particle quantum states. Finally, we introduce the scheme of modulation and demodulation on classical fields with PPSs.

A. Pseudorandom sequences and their properties

As far as we know, orthogonal pseudorandom sequences have been widely applied to CDMA communication technology as a way to distinguish different users [6, 7]. A set of pseudorandom sequences is generated by using a shift register guided by a Galois Field $GF(p)$, that satisfies orthogonal, closure and balance properties. The orthogonal property ensures that sequences of the set are independent and distinguished each other with an excellent correlation property. The closure property ensures that any linear combination of the sequences remains in the same set. The balance property ensures that the occurrence rate of all non-zero-element equals with each other, and the the number of zero-elements is exactly one less than the other elements.

One famous generator of pseudorandom sequences is Linear Feedback Shift Register

(LFSR), which could produce a maximal period sequence, called m-sequence [7]. We consider an m-sequence of period $N - 1$ ($N = p^s$) generated by a primitive polynomial of degree s over $\text{GF}(p)$. Since the correlation between different shifts of an m-sequence is almost zero, they can be used as different codes with their excellent correlation property. In this regard, the set of $N - 1$ m-sequences of length $N - 1$ could be obtained by cyclic shifting of a single m-sequence.

In this paper, we employ pseudorandom phase sequences (PPSs) with 4-ary phase shift modulation, which is a well-known modulation format in wireless and optical communications, including Orthogonal Quadrature Phase Shift Keying (O-QPSK) and Minimum Phase Shift Keying (MSK) [9]. We first propose a scheme to generate a PPS set $\Xi = \{\lambda^{(0)}, \lambda^{(1)}, \dots, \lambda^{(N-1)}\}$ over $\text{GF}(4)$ [10]. $\lambda^{(0)}$ is an all-0 sequence and other sequences can be generated by using the method as follows:

- (1) given a primitive polynomial of degree s over $\text{GF}(4)$, a base sequence of a length $4^s - 1$ is generated by using LFSR;
- (2) other sequences are obtained by cyclic shifting of the base sequence;
- (3) by adding a zero-element to the end of each sequence, the occurrence rates of all elements in all sequences are equal with each other;
- (4) mapping the elements of the sequences to $[0, 2\pi]$: 0 mapping 0, 1 mapping $\pi/2$, 2 mapping π , and 3 mapping $3\pi/2$.

Further, we define a map $f : \lambda \rightarrow e^{i\lambda}$ on the set of Ξ , and obtain a new sequence set $\Omega = \{\varphi^{(j)} \mid \varphi^{(j)} = e^{i\lambda^{(j)}}, j = 0, \dots, N - 1\}$. According to the properties of m-sequence, we can obtain following properties of the set Ω , (1) the closure property: the product of any sequences remains in the same set; (2) the balance property: in exception to $\varphi^{(0)}$, any sequence of the set Ω satisfy

$$\sum_{k=1}^N e^{i\theta} \varphi_k^{(j)} = \sum_{k=1}^N e^{i(\lambda_k^{(j)} + \theta)} = 0, \forall \theta \in \mathbb{R}; \quad (1)$$

- (3) the orthogonal property: any two sequences satisfy the following normalized correlation

$$\begin{aligned} E(\varphi^{(i)}, \varphi^{(j)}) &= \frac{1}{N} \sum_{k=1}^N \varphi_k^{(i)} \varphi_k^{(j)*} \\ &= \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}. \end{aligned} \quad (2)$$

In fact, the map f corresponds to the modulation of PPSs of Ω on classical fields. According to the properties above, the classical fields modulated with different PPSs become independent and distinguishable.

B. Similarities between classical field and single-particle quantum states

We note the similarities between Maxwell equation and Schrödinger equation. In fact, some properties of quantum information are wave properties, where the wave need not be a quantum wave [3]. Analogous to quantum states, classical fields also obey a superposition principle, and could be transformed to any superposition state by unitary transformations. Those analogous properties made the simulation of quantum states using polarization or transverse modes of classical fields possible [11–13].

We first consider two orthogonal modes (polarization or transverse), which are denoted by $|0\rangle$ and $|1\rangle$ respectively, as the classical simulation of quantum bits (qubits) $|0\rangle$ and $|1\rangle$ [3, 11].

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (3)$$

Thus, any quantum state $|\varphi\rangle = \alpha|0\rangle + \beta|1\rangle$ can be simulated by a corresponding classical mode superposition field, as follows

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, |\alpha|^2 + |\beta|^2 = 1, (\alpha, \beta \in \mathbb{C}). \quad (4)$$

Obviously, all the mode superposition fields could span a Hilbert space, where we can perform unitary transformations to transform the mode state. For example, the unitary transformation $U(\chi, \theta)$ is defined

$$U(\chi, \theta) = e^{i\chi(\sigma_x \cos \theta + \sigma_y \sin \theta)} = \begin{pmatrix} \cos \chi & -ie^{i\theta} \sin \chi \\ ie^{-i\theta} \sin \chi & \cos \chi \end{pmatrix}, \quad (5)$$

where σ_x, σ_y are Pauli matrices. The modes $|0\rangle$ and $|1\rangle$ can be transformed to mode superposition by using $U(\chi, \theta)$, respectively, as follows

$$\begin{aligned} U(\chi, \theta)|0\rangle &= \cos \chi |0\rangle + ie^{i\theta} \sin \chi |1\rangle, \\ U(\chi, \theta)|1\rangle &= \cos \chi |1\rangle - ie^{-i\theta} \sin \chi |0\rangle. \end{aligned} \quad (6)$$

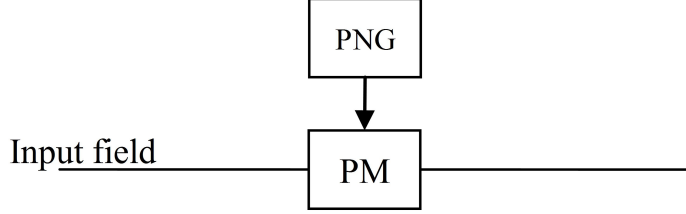


FIG. 1: The PPS encoding scheme for one input field, where PNG denotes the pseudorandom number generator and PM denotes the phase modulator.

Now, we consider some devices with one input and two outputs, such as beam or mode splitters, which split one input field $|\psi_{in}\rangle = \alpha|0\rangle + \beta|1\rangle$ into two output fields $|\psi_{out}^{(a)}\rangle$ and $|\psi_{out}^{(b)}\rangle$. For the case of beam splitters, the output fields are $|\psi_{out}^{(a)}\rangle = C_a(\alpha|0\rangle + \beta e^{i\phi_a}|1\rangle)$ and $|\psi_{out}^{(b)}\rangle = C_b(\alpha|0\rangle + \beta e^{i\phi_b}|1\rangle)$ with an arbitrary power ratio $|C_a|^2 : |C_b|^2$ between the output beams, where $\phi_{a,b}$ are the additional phases due to the splitter. For the case of mode splitters, the output fields are $|\psi_{out}^{(a)}\rangle = \alpha e^{i\phi_a}|0\rangle$ and $|\psi_{out}^{(b)}\rangle = \beta e^{i\phi_b}|1\rangle$, where $\phi_{a,b}$ are also the additional phases. Conversely, the devices can act as beam or mode combiners in which beams or modes from two inputs are combined into one output.

C. Modulation and demodulation on classical fields with pseudorandom phase sequences

We first consider the modulation process on a classical field with a PPS. Similar to O-QPSK system, chosen a PPS $\lambda^{(i)}$ in the set of Ξ , the phase of the field could be modulated by a phase modulator (PM) that controlled by a pseudorandom number generator (PNG), the scheme is shown in Fig. 1. If the input is a single-mode field, it could be transformed to mode superposition by performing a unitary transformation after the modulation.

Then we consider the quadrature demodulation process of a modulated classical field. Quadrature demodulation is a coherent detection process that allows the simultaneous measurement of conjugate quadrature components via homodyning the emerging beams with the input and reference fields by using a balanced beam splitting, where the reference field is modulated with a PPS $\lambda^{(r)}$. The differenced signals of two output detectors are then integrated and sampled to yield the decision variable. We can express the demodulation

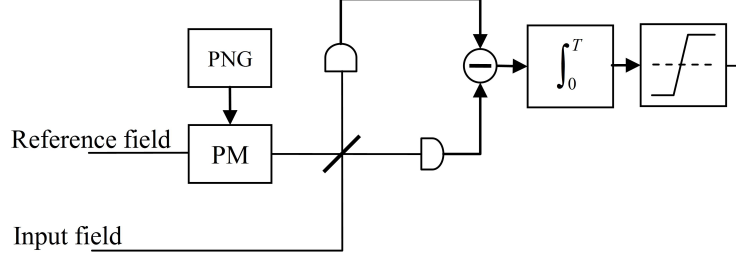


FIG. 2: The PPS quadrature demodulation scheme for one input field.

process in mathematical form

$$\begin{aligned}
 I(\lambda^{(i)}, \lambda^{(r)}) &= \frac{1}{N} \sum_{k=1}^N \cos(\lambda_k^{(i)} - \lambda_k^{(r)}) \\
 &= \begin{cases} 1, & i = r \\ 0, & i \neq r \end{cases},
 \end{aligned} \tag{7}$$

where $\lambda_k^{(i)}, \lambda_k^{(r)}$ are the PPSs of the input and reference fields respectively. The output decision variable is 1 if and only if $\lambda_k^{(i)}, \lambda_k^{(r)}$ are equal; otherwise the output decision variable is 0. The results are guaranteed by the properties of PPSs. If the input is a single-mode field, the scheme shown in Fig. 2 is employed to perform quadrature demodulation. Otherwise the scheme shown in Fig. 3 is used, in which the input field $|\psi_i\rangle = e^{i\lambda^{(i)}}(\alpha_i|0\rangle + \beta_i|1\rangle)$ is first splitted into two fields $\alpha_i e^{i\lambda^{(i)}}|0\rangle$ and $\beta_i e^{i\lambda^{(i)}}|1\rangle$, and two coherent detection processes are then performed on the two fields respectively. Noteworthily, the modes of the reference fields must be consistent with the two output fields. Thus there are two output decision variables $\tilde{\alpha}$ and $\tilde{\beta}$, which correspond to the modes $|0\rangle$ and $|1\rangle$, respectively. We define $(\tilde{\alpha}, \tilde{\beta})$ as the mode status. Besides the quadrature demodulation, we can also easily measure the amplitudes of the fields $|\alpha_i|, |\beta_i|$ after mode spitting in the scheme.

III. CLASSICAL SIMULATION OF MULTIPARTICLE QUANTUM STATES

We discuss simulation of multiparticle quantum states using classical fields modulated with PPSs in this section. We first demonstrate that n classical fields modulated with n different PPSs could constitute a similar 2^n -dimensional Hilbert space that contains a tensor product structure. Besides, by performing quadrature demodulation scheme, we could obtain the mode status matrix of the simulating classical fields, based on which we

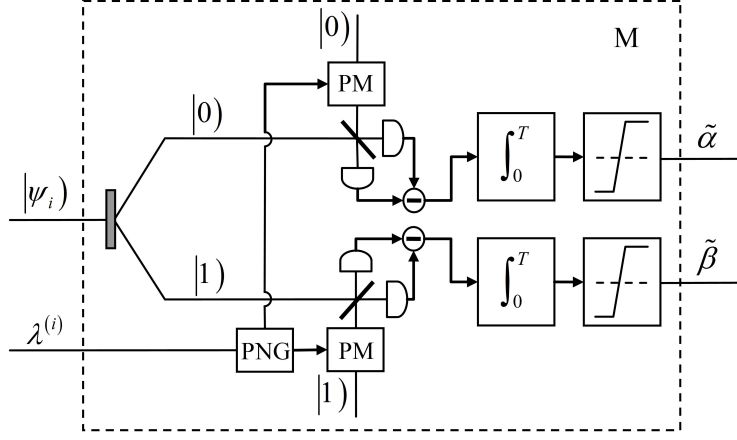


FIG. 3: The PPS quadrature demodulation scheme for one field with two orthogonal modes, where the gray block denote the mode splitter.

propose a sequence permutation mechanism to reconstruct the simulated quantum states. The classical simulation of some typical quantum states is discussed, including product state, Bell states, GHZ state and W state.

A. Classical fields modulated with pseudorandom phase sequences and their tensor product structure

For convenience, here we consider two classical fields modulated with PPSs and their tensor product structure. Chosen two PPSs of $\lambda^{(a)}$ and $\lambda^{(b)}$ from the set Ξ , any two fields modulated with the PPSs can be expressed as follows,

$$\begin{aligned} |\psi_a\rangle &= e^{i\lambda^{(a)}} (\alpha_a |0\rangle + \beta_a |1\rangle), \\ |\psi_b\rangle &= e^{i\lambda^{(b)}} (\alpha_b |0\rangle + \beta_b |1\rangle). \end{aligned} \quad (8)$$

We define the inner product of two fields $|\psi_a\rangle$ and $|\psi_b\rangle$ as follows,

$$\langle \psi_a | \psi_b \rangle = \frac{1}{N} \sum_{k=1}^N e^{i(\lambda_k^{(a)} - \lambda_k^{(b)})} (\alpha_b^* \alpha_a + \beta_b^* \beta_a). \quad (9)$$

According to the properties of the PPSs, we can easily obtain

$$\langle \psi_a | \psi_b \rangle = \begin{cases} 1, & a = b \\ 0, & a \neq b \end{cases}, \quad (10)$$

which shows that two fields modulated with two different PPSs are orthogonal. The orthogonal property supports to construct the tensor product structure of multiple fields.

Assume two Hilbert spaces w and v spanned by the states $|\psi_a\rangle$ and $|\psi_b\rangle$, any linear combinations of the elements in the direct product space of $w \otimes v$ remain in the same space. We define the two orthogonal modes modulated with the PPS $\lambda^{(a)}$ as the orthonormal bases of the space of w , expressed as $|0_a\rangle \equiv e^{i\lambda^{(a)}}|0\rangle$ and $|1_a\rangle \equiv e^{i\lambda^{(a)}}|1\rangle$. Using the same notion, the orthonormal bases of the space of v are expressed as $|0_b\rangle \equiv e^{i\lambda^{(b)}}|0\rangle$ and $|1_b\rangle \equiv e^{i\lambda^{(b)}}|1\rangle$. The four orthonormal bases are thus independent and distinguishable. Then the orthonormal bases of the direct product space of $w \otimes v$ can be expressed as $\{|0_a\rangle \otimes |0_b\rangle, |0_a\rangle \otimes |1_b\rangle, |1_a\rangle \otimes |0_b\rangle, |1_a\rangle \otimes |1_b\rangle\}$. Further, we can obtain the following tensor product properties,

(1) for any scalar z , the elements $|\psi_a\rangle, |\psi_b\rangle$ in the spaces of w and v , respectively, satisfy

$$z(|\psi_a\rangle \otimes |\psi_b\rangle) = (z|\psi_a\rangle) \otimes |\psi_b\rangle = |\psi_a\rangle \otimes (z|\psi_b\rangle); \quad (11)$$

(2) in the space of $w \otimes v$, the direct product of the combinations of elements is equal to the combination of the direct products of elements,

$$\begin{aligned} (|\psi_a\rangle + |\psi'_a\rangle) \otimes (|\psi_b\rangle + |\psi'_b\rangle) &= |\psi_a\rangle \otimes |\psi_b\rangle + |\psi_a\rangle \otimes |\psi'_b\rangle \\ &\quad + |\psi'_a\rangle \otimes |\psi_b\rangle + |\psi'_a\rangle \otimes |\psi'_b\rangle. \end{aligned} \quad (12)$$

Using the same notion, we can construct a 2^n -dimensional direct product space of $|\psi\rangle^{\otimes n} \equiv |\psi_1\rangle \otimes \dots \otimes |\psi_n\rangle$ by using the mode superposition of n classical fields $|\psi_1\rangle, \dots, |\psi_n\rangle$ modulated with n PPSs.

Quantum entanglement is only defined for Hilbert spaces that have a rigorous tensor product structure in terms of subsystems. Here we construct a similar structure of multiple classical fields, which is the basis of efficient classical simulation of quantum entanglement.

B. Reconstruction of quantum states based on the simulating classical fields

We have discussed quadrature demodulation process in Sec. II C. Here we discuss how to reconstruct quantum state based on the simulating classical fields with the help of quadrature demodulation.

First, we consider the general form of n classical fields modulated with PPSs $\{\lambda^{(1)}, \dots, \lambda^{(n)}\}$ chosen from the set Ξ ,

$$\begin{aligned} |\psi_1\rangle &= \frac{1}{C_1} \left[\left(e^{i\lambda^{(a)}} + \dots + e^{i\lambda^{(b)}} \right) |0\rangle + \left(e^{i\lambda^{(c)}} + \dots + e^{i\lambda^{(d)}} \right) |1\rangle \right], \\ &\dots\dots\dots \\ |\psi_n\rangle &= \frac{1}{C_n} \left[\left(e^{i\lambda^{(e)}} + \dots + e^{i\lambda^{(f)}} \right) |0\rangle + \left(e^{i\lambda^{(g)}} + \dots + e^{i\lambda^{(h)}} \right) |1\rangle \right], \end{aligned} \quad (13)$$

where C_i are the normalized coefficients and a, \dots, h are the sequence numbers. It is noteworthy that although multiple PPSs are superimposed on both modes of the fields, all of the PPSs could be demodulated and discriminated by performing the quadrature demodulation introduced in Sec. II C, which has already been verified by many actual communication systems [9].

Now we propose a scheme, as shown in Fig. 4, to perform the quadrature demodulation introduced in Sec. II C. In the scheme, quadrature demodulations are performed on each field, in which the reference PPSs are ergodic on $\{\lambda^{(1)}, \dots, \lambda^{(n)}\}$. Thus a mode status matrix $M(\tilde{\alpha}_i^j, \tilde{\beta}_i^j)$, as shown in Fig. 5, could be obtained by performing n quadrature demodulations on the n classical fields. Of the matrix $M(\tilde{\alpha}_i^j, \tilde{\beta}_i^j)$, each element is the mode status of the i th classical field when the reference PPS is $\lambda^{(j)}$. The element takes one of four possible values: $(1, 0)$, $(0, 1)$, $(1, 1)$ or 0 , denote that the PPS $\lambda^{(j)}$ is modulated on mode $|0\rangle$ of the i th classical field, on mode $|1\rangle$, on both $|0\rangle$ and $|1\rangle$, on neither $|0\rangle$ nor $|1\rangle$, respectively. It is noteworthy that different modulation of the n classical fields correspond to different mode status matrixes, and vice versa. Thus we obtain a one-to-one correspondence relationship between the n classical fields and the mode status matrix. Besides, further discussion will show that structure of quantum states and quantum entanglement could be revealed in the mode status matrix, which means that a correspondence could also be obtained between the mode status matrix and quantum states. Thus we treat the mode status matrix as a bridge to connect the simulating fields and the quantum states.

Now we focus on the correspondence between the mode status matrix and quantum states, and consider how to reconstruct quantum states based on the mode status matrix. We first transform the matrix $M(\tilde{\alpha}_i^j, \tilde{\beta}_i^j)$ to a block diagonal matrix by permuting the fields and the sequences, namely the rows and columns in the matrix, respectively, and obtain a matrix

expressed as similar to

$$M \left(\tilde{\alpha}_i^j, \tilde{\beta}_i^j \right) = \begin{pmatrix} M_1^1 & & & & & \\ & M_2^2 & M_2^3 & & & \\ & M_3^2 & M_3^3 & & & \\ & & & M_4^4 & M_4^5 & M_4^6 \\ & & & M_5^4 & M_5^5 & M_5^6 \\ & & & M_6^4 & M_6^5 & M_6^6 \\ & & & & & \ddots \end{pmatrix}. \quad (14)$$

The structure of the matrix could clearly reveal the structure of the simulated quantum state. Each diagonal block of the matrix denotes one unreduced subsystem of the simulated state $|\Psi\rangle$. Thus the state $|\Psi\rangle$ could be expressed as the direct product of the unreduced states $|\Psi_b\rangle$, as follows

$$|\Psi\rangle = \prod_{b=1}^m |\Psi_b\rangle, \quad (15)$$

where m denotes the number of the matrix blocks.

Now we consider how to reconstruct each $|\Psi_b\rangle$ based on each submatrix, respectively. As each submatrix is irreducible, each $|\Psi_b\rangle$ corresponds to an entangled state. More important, different entanglement structures correspond to different structures of the unreduced submatrix. Thus a correspondence relationship could be obtained between quantum entanglement and the unreduced mode status matrix. In order to reconstruct the quantum entanglement state, an ergodic ensemble of PPSs is required to obtain all possible base states. Thus we propose a sequence permutation mechanism to reconstruct each $|\Psi_b\rangle$ based on each submatrix, which is one of the simplest mechanisms for sequence ergodic ensemble. Assumed $|\Psi_b\rangle$ contains l classical fields with l PPSs, namely the corresponding submatrix contains l rows and l columns, the sequence permutation is arranged as

$$R_1 = \{\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(l)}\}, R_2 = \{\lambda^{(2)}, \lambda^{(3)}, \dots, \lambda^{(1)}\}, \dots, R_l = \{\lambda^{(l)}, \lambda^{(1)}, \dots, \lambda^{(l-1)}\}. \quad (16)$$

In the mechanism, each R_r corresponds to one selection from the unreduced submatrix. We obtain a direct product of l items for each R_r , and the simulated quantum state is the superposition of the l product items. Therefore we obtain

$$|\Psi_b\rangle = \frac{1}{C} \sum_{r=1}^l \prod_{i=1}^l \left(\tilde{\alpha}_i^{j[R_r^i]} |0\rangle + \tilde{\beta}_i^{j[R_r^i]} |1\rangle \right), \quad (17)$$

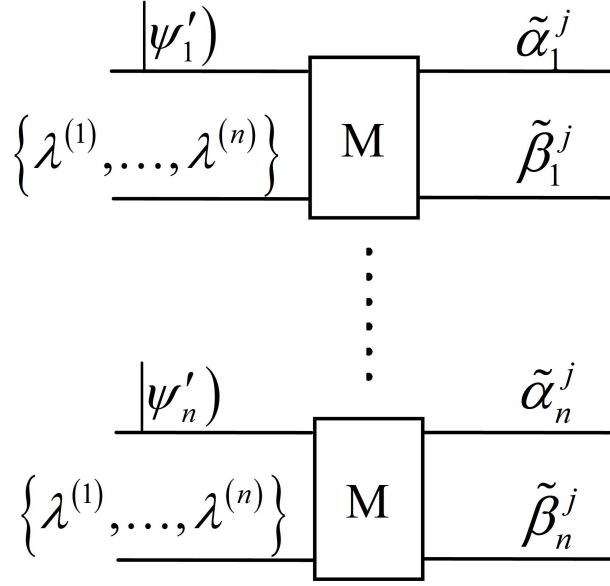


FIG. 4: The PPS quadrature demodulation scheme for multiple input fields, where the M block is shown in Fig. 3.

where C is the normalized coefficient, $\left(\tilde{\alpha}_i^{j[R_r^i]}, \tilde{\beta}_i^{j[R_r^i]}\right)$ is the mode status obtained from the submatrix $M\left(\tilde{\alpha}_i^j, \tilde{\beta}_i^j\right)$, where $j[R_r^i]$ denotes the sequence number of the i th sequence in R_r .

It is noteworthy that the mechanism we proposed above is one of the feasible ways to reconstruct the simulated quantum state based on the unreduced mode status matrix. Other mechanisms might also work, as long as a sequence ergodic ensemble is obtained in the mechanism. The sequence permutation mechanism above could successfully reconstruct many quantum states, including the product states, Bell states, GHZ states and W states. We will discuss the related contents in next subsection.

C. Classical simulation of several typical quantum states

In this subsection, we discuss classical simulation of several typical quantum states, including product state, Bell states, GHZ state and W state.

$$\begin{array}{cccc}
& \lambda^{(1)} & \lambda^{(2)} & \dots & \lambda^{(n)} \\
\begin{array}{c} |\psi'_1\rangle \\ |\psi'_2\rangle \\ \vdots \\ |\psi'_n\rangle \end{array} & \begin{pmatrix} M_1^1 & M_1^2 & \dots & M_1^n \\ M_2^1 & M_2^2 & \dots & M_2^n \\ \dots & \dots & \dots & \dots \\ M_n^1 & M_n^2 & \dots & M_n^n \end{pmatrix}
\end{array}$$

FIG. 5: The mode status matrix related to the fields and PPSs, where M_i^j is the element of $M(\tilde{\alpha}_i^j, \tilde{\beta}_i^j)$ for the i th classical field and the reference PPS $\lambda^{(j)}$.

1. Classical simulation of product state

First, we discuss classical simulation of n quantum product state. The simulation fields are shown as follows

$$\begin{aligned}
|\psi_1\rangle &= \frac{e^{i\lambda^{(1)}}}{\sqrt{2}} (|0\rangle + |1\rangle), \\
&\dots\dots\dots \\
|\psi_n\rangle &= \frac{e^{i\lambda^{(n)}}}{\sqrt{2}} (|0\rangle + |1\rangle).
\end{aligned} \tag{18}$$

By employing the scheme as shown in Fig. 4, we obtain the mode status matrix

$$M(\tilde{\alpha}_i^j, \tilde{\beta}_i^j) = \begin{pmatrix} (1,1) & & 0 \\ & \ddots & \\ 0 & & (1,1) \end{pmatrix}, \tag{19}$$

which demonstrates that each classical field is the superposition of two orthogonal modes and no entanglement is involved. According to Eq. (15), we obtain

$$\begin{aligned}
|\Psi\rangle &= \frac{1}{2^{n/2}} (|0\rangle + |1\rangle) \otimes \dots \otimes (|0\rangle + |1\rangle) \\
&= \frac{1}{2^{n/2}} (|0\dots 0\rangle + |0\dots 1\rangle + \dots + |1\dots 1\rangle),
\end{aligned} \tag{20}$$

where $|q_1 \dots q_n\rangle \equiv |q_1\rangle \otimes \dots \otimes |q_n\rangle$, ($q_i = 0$ or 1).

2. Classical simulation of Bell states

Now we discuss classical simulation of one of the four Bell states $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|0_a\rangle|0_b\rangle + |1_a\rangle|1_b\rangle)$, which contains two classical fields as follows

$$\begin{aligned} |\psi_a\rangle &= \frac{1}{\sqrt{2}} \left(e^{i\lambda^{(a)}} |0\rangle + e^{i\lambda^{(b)}} |1\rangle \right), \\ |\psi_b\rangle &= \frac{1}{\sqrt{2}} \left(e^{i\lambda^{(b)}} |0\rangle + e^{i\lambda^{(a)}} |1\rangle \right). \end{aligned} \quad (21)$$

By employing the scheme as shown in Fig. 4, we obtain the mode status matrix

$$M \left(\tilde{\alpha}_i^j, \tilde{\beta}_i^j \right) = \begin{pmatrix} (1, 0) & (0, 1) \\ (0, 1) & (1, 0) \end{pmatrix}. \quad (22)$$

We note that in this case, the mode status matrix is irreducible, which corresponds to an entanglement state. According to the sequence permutation mechanism, we obtain that $R_1 = \{\lambda^{(a)}, \lambda^{(b)}\}$ and $R_2 = \{\lambda^{(b)}, \lambda^{(a)}\}$. Based on the mode status matrix, for the selection of R_1 , we obtain $|0\rangle \otimes |0\rangle$; for the selection of R_2 , we obtain $|1\rangle \otimes |1\rangle$. If we randomly choose one selection between R_1 and R_2 , we could randomly obtain one result between $|0\rangle \otimes |0\rangle$ and $|1\rangle \otimes |1\rangle$, which is similar with the case of quantum measurement for the Bell state $|\Psi^+\rangle$. We could reconstruct the state based on the mode status matrix

$$\begin{aligned} |\Psi^+\rangle &= \frac{1}{\sqrt{2}} \sum_{r=1}^2 \left[\left(\tilde{\alpha}_a^{j[R_r^a]} |0\rangle + \tilde{\beta}_a^{j[R_r^a]} |1\rangle \right) \otimes \left(\tilde{\alpha}_b^{j[R_r^b]} |0\rangle + \tilde{\beta}_b^{j[R_r^b]} |1\rangle \right) \right] \\ &= \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle). \end{aligned} \quad (23)$$

In quantum mechanics, another Bell state $|\Phi^+\rangle$ could be obtained from $|\Psi^+\rangle$ by performing the unitary transformation $\sigma_x : |0\rangle \leftrightarrow |1\rangle$ on one of the particles. Using the same method, we perform an unitary transformation on $|\psi_b\rangle$ to flip its modes $|0\rangle \leftrightarrow |1\rangle$. Thus we obtain two classical fields as follows

$$\begin{aligned} |\psi_a\rangle &= \frac{1}{\sqrt{2}} \left(e^{i\lambda^{(a)}} |0\rangle + e^{i\lambda^{(b)}} |1\rangle \right), \\ |\psi_b\rangle &= \frac{1}{\sqrt{2}} \left(e^{i\lambda^{(b)}} |1\rangle + e^{i\lambda^{(a)}} |0\rangle \right). \end{aligned} \quad (24)$$

By employing the scheme as shown in Fig. 4, we obtain the mode status matrix

$$M \left(\tilde{\alpha}_i^j, \tilde{\beta}_i^j \right) = \begin{pmatrix} (1, 0) & (0, 1) \\ (1, 0) & (0, 1) \end{pmatrix}. \quad (25)$$

According to the sequence permutation mechanism, here we obtain $R_1 = \{\lambda^{(a)}, \lambda^{(b)}\}$ and $R_2 = \{\lambda^{(b)}, \lambda^{(a)}\}$ again. As the mode status matrix is different, for R_1 , the result turns to be $|0\rangle \otimes |1\rangle$; for R_2 , we obtain $|1\rangle \otimes |0\rangle$. If we randomly choose one selection between R_1 and R_2 , we could also randomly obtain one result between $|0\rangle \otimes |1\rangle$ and $|1\rangle \otimes |0\rangle$, which is similar with the case of quantum measurement for the Bell state $|\Phi^+\rangle$. We could reconstruct the state

$$\begin{aligned} |\Phi^+\rangle &= \frac{1}{\sqrt{2}} \sum_{r=1}^2 \left[\left(\tilde{\alpha}_a^{j[R_r^a]} |0\rangle + \tilde{\beta}_a^{j[R_r^a]} |1\rangle \right) \otimes \left(\tilde{\alpha}_b^{j[R_r^b]} |0\rangle + \tilde{\beta}_b^{j[R_r^b]} |1\rangle \right) \right] \\ &= \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle). \end{aligned} \quad (26)$$

For other two Bell states $|\Psi^-\rangle$ and $|\Phi^-\rangle$, they could be obtained from $|\Psi^+\rangle$ and $|\Phi^+\rangle$ by using a π phase transformation. However, they could not be distinguished from $|\Psi^+\rangle$ and $|\Phi^+\rangle$ unless an orthogonal projection measurement is performed [8].

3. Classical simulation of GHZ state

For tripartite systems there are only two different classes of genuine tripartite entanglement, the GHZ class and the W class [14, 15]. First we discuss the classical simulation of GHZ state $|\Psi_{GHZ}\rangle = \frac{1}{\sqrt{2}} (|0_a\rangle |0_b\rangle |0_c\rangle + |1_a\rangle |1_b\rangle |1_c\rangle)$, which contains three classical fields as follows

$$\begin{aligned} |\psi_a\rangle &= \frac{1}{\sqrt{2}} \left(e^{i\lambda^{(a)}} |0\rangle + e^{i\lambda^{(b)}} |1\rangle \right), \\ |\psi_b\rangle &= \frac{1}{\sqrt{2}} \left(e^{i\lambda^{(b)}} |0\rangle + e^{i\lambda^{(c)}} |1\rangle \right), \\ |\psi_c\rangle &= \frac{1}{\sqrt{2}} \left(e^{i\lambda^{(c)}} |0\rangle + e^{i\lambda^{(a)}} |1\rangle \right). \end{aligned} \quad (27)$$

Performing the scheme as shown in Fig. 4, we obtain the mode status matrix

$$M \left(\tilde{\alpha}_i^j, \tilde{\beta}_i^j \right) = \begin{pmatrix} (1,0) & (0,1) & 0 \\ 0 & (1,0) & (0,1) \\ (0,1) & 0 & (1,0) \end{pmatrix}. \quad (28)$$

According to the sequence permutation mechanism, we obtain that $R_1 = \{\lambda^{(a)}, \lambda^{(b)}, \lambda^{(c)}\}$, $R_2 = \{\lambda^{(b)}, \lambda^{(c)}, \lambda^{(a)}\}$ and $R_3 = \{\lambda^{(c)}, \lambda^{(a)}, \lambda^{(b)}\}$. Based on the mode status matrix, for the selection of R_1 , we obtain $|0\rangle \otimes |0\rangle \otimes |0\rangle$; for the selection of R_2 , we obtain $|1\rangle \otimes |1\rangle \otimes |1\rangle$;

for the selection of R_3 , we obtain nothing. Thus we could reconstruct the state based on the mode status matrix

$$\begin{aligned} |\Psi_{GHZ}\rangle &= \frac{1}{\sqrt{2}} \sum_{r=1}^3 \left[\left(\tilde{\alpha}_a^{j[R_r^a]} |0\rangle + \tilde{\beta}_a^{j[R_r^a]} |1\rangle \right) \otimes \left(\tilde{\alpha}_b^{j[R_r^b]} |0\rangle + \tilde{\beta}_b^{j[R_r^b]} |1\rangle \right) \otimes \left(\tilde{\alpha}_c^{j[R_r^c]} |0\rangle + \tilde{\beta}_c^{j[R_r^c]} |1\rangle \right) \right] \\ &= \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle). \end{aligned} \quad (29)$$

4. Classical simulation of W state

Then we discuss the classical simulation of W state,

$$|\Psi_W\rangle = \frac{1}{\sqrt{3}} (|1_a\rangle |0_b\rangle |0_c\rangle + |0_a\rangle |1_b\rangle |0_c\rangle + |0_a\rangle |0_b\rangle |1_c\rangle), \quad (30)$$

which contains three classical fields as follows

$$\begin{aligned} |\psi_a\rangle &= \frac{1}{\sqrt{3}} \left(e^{i\lambda^{(a)}} |1\rangle + e^{i\lambda^{(b)}} |0\rangle + e^{i\lambda^{(c)}} |0\rangle \right), \\ |\psi_b\rangle &= \frac{1}{\sqrt{3}} \left(e^{i\lambda^{(a)}} |1\rangle + e^{i\lambda^{(b)}} |0\rangle + e^{i\lambda^{(c)}} |0\rangle \right), \\ |\psi_c\rangle &= \frac{1}{\sqrt{3}} \left(e^{i\lambda^{(a)}} |1\rangle + e^{i\lambda^{(b)}} |0\rangle + e^{i\lambda^{(c)}} |0\rangle \right), \end{aligned} \quad (31)$$

It is noteworthy that the three classical fields could be produced from one single field by using two beam splitters, which is quite similar with the generation of W state in quantum mechanics. Performing the same scheme, we obtain the mode status matrix

$$M \left(\tilde{\alpha}_i^j, \tilde{\beta}_i^j \right) = \begin{pmatrix} (0, 1) & (1, 0) & (1, 0) \\ (0, 1) & (1, 0) & (1, 0) \\ (0, 1) & (1, 0) & (1, 0) \end{pmatrix}. \quad (32)$$

According to the sequence permutation mechanism, we obtain that $R_1 = \{\lambda^{(a)}, \lambda^{(b)}, \lambda^{(c)}\}$, $R_2 = \{\lambda^{(b)}, \lambda^{(c)}, \lambda^{(a)}\}$ and $R_3 = \{\lambda^{(c)}, \lambda^{(a)}, \lambda^{(b)}\}$ again. Based on the mode status matrix, we obtain $|1\rangle \otimes |0\rangle \otimes |0\rangle$, $|0\rangle \otimes |0\rangle \otimes |1\rangle$, $|0\rangle \otimes |1\rangle \otimes |0\rangle$ for the selection of R_1 , R_2 , R_3 , respectively. We find an interesting fact that when we obtain the state $|1\rangle$ of the first field, R_1 must be selected, thus only the $|0\rangle \otimes |0\rangle$ state could be obtained from the other two fields; otherwise when we obtain the state $|0\rangle$ of the first field, the selection could be R_2 or R_3 , thus the state of $|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle$ could be obtained from the other two fields. This fact is quite

similar with the case of quantum measurement and the collapse phenomenon for W state in quantum mechanics. We could reconstruct the state based on the mode status matrix

$$\begin{aligned} |\Psi_W\rangle &= \frac{1}{\sqrt{3}} \sum_{r=1}^3 \left[\left(\tilde{\alpha}_a^{j[R_r^a]} |0\rangle + \tilde{\beta}_a^{j[R_r^a]} |1\rangle \right) \otimes \left(\tilde{\alpha}_b^{j[R_r^b]} |0\rangle + \tilde{\beta}_b^{j[R_r^b]} |1\rangle \right) \otimes \left(\tilde{\alpha}_c^{j[R_r^c]} |0\rangle + \tilde{\beta}_c^{j[R_r^c]} |1\rangle \right) \right] \\ &= \frac{1}{\sqrt{3}} (|100\rangle + |010\rangle + |001\rangle). \end{aligned} \quad (33)$$

IV. GENERALIZATION OF THE SIMULATION AND SOME DISCUSSIONS ON EFFICIENCY

In this section, we first propose a generalization of our simulation to the case of an arbitrary quantum state, then the efficiency of our simulation is discussed. Besides, for better understanding our simulation, the PPSs in the cases of modulating two and three classical fields is illustrated.

A. Generalization of the simulation

Here we propose a generalization of our simulation to the case of an arbitrary quantum state. In Sec. III C, classical simulation of several typical quantum states has been introduced, including product state, Bell states, GHZ state, and W state, based on which we assume that any quantum state of $n - 1$ particles could be successfully simulated. Here we consider the classical simulation of an arbitrary quantum state of n particles, $|\Psi_n\rangle$. We first transform $|\Psi_n\rangle$ to an equivalent expression, $|\Psi_n\rangle = |\Phi_{n-1}\rangle |0\rangle_n + |\Theta_{n-1}\rangle |1\rangle_n$, where $|\Phi_{n-1}\rangle$ and $|\Theta_{n-1}\rangle$ denote two quantum states of $n - 1$ particles. According to the assumption, we obtain that the quantum state $|\Phi_{n-1}\rangle$ could be simulated by $n - 1$ classical fields $|\varphi_1\rangle, \dots, |\varphi_{n-1}\rangle$ modulated with $n - 1$ PPSs $\{\lambda^{(1)}, \dots, \lambda^{(n-1)}\}$, and the quantum state $|\Theta_{n-1}\rangle$ could be simulated by $n - 1$ classical fields $|\vartheta_1\rangle, \dots, |\vartheta_{n-1}\rangle$ modulated with $n - 1$ PPSs $\{\lambda^{(1)}, \dots, \lambda^{(n-2)}, \lambda^{(n)}\}$.

Further, we consider the classical simulation of the quantum state $|\Psi_n\rangle$, which is the superposition of the classical simulation of $|\Phi_{n-1}\rangle \otimes |0\rangle_n$ and the classical simulation of $|\Theta_{n-1}\rangle \otimes |1\rangle_n$,

$$|\psi_1\rangle = \frac{1}{C_1} [|\varphi_1\rangle + |\vartheta_1\rangle], \quad (34)$$

$$\begin{aligned}
& \dots\dots \\
|\psi_{n-1}\rangle &= \frac{1}{C_{n-1}}[|\varphi_{n-1}\rangle + |\vartheta_{n-1}\rangle], \\
|\psi_n\rangle &= \frac{1}{C_n} \left[e^{i\lambda^{(n)}} |0\rangle + e^{i\lambda^{(n-1)}} |1\rangle \right].
\end{aligned}$$

Thus by only adding one classical field and one PPS, we obtain the classical simulation of the quantum state $|\Psi_n\rangle$ based on the assumed classical simulation of $|\Phi_{n-1}\rangle$ and $|\Theta_{n-1}\rangle$. Using the principle of induction, we could provide classical simulation of any quantum state. Therefore we successfully generalize our simulation.

B. Efficiency of the classical simulation

For a long time, researchers have used classical fields to simulate quantum states and quantum computation. In these researches, multiple qubits are distinguished by different degrees of freedom, such as optical modes or space positions. However, as no tensor product structure is obtained in these classical simulations, each quantum base state needs independent degree of freedom to simulate. In quantum mechanics, the number of quantum base states grows exponentially with the number of quantum particles. Therefore the classical simulation require resources that also grow exponentially with the number of simulated quantum particles, which is not efficient.

In this paper, we utilize the properties of PPSs to distinguish classical fields that are even overlapped in same space and time. A 2^n -dimensional Hilbert space which contains tensor product structure is spanned by n classical fields modulated with PPSs. In our scheme, the resources required are classical fields modulated with PPSs instead of optical/space modes. One classical field modulated with one PPS can simulate one quantum particle. It means that the amount of classical fields and PPSs grows linearly with the number of quantum particles. According to the m-sequence theory, the number of PPSs in the set Ξ equals to the length of sequences, which means that the time resource (the length of sequence) required also grows linearly with the number of the particles. Based on the analysis above, we conclude that one can efficiently simulate quantum entanglement with linearly growing resources by using our scheme.

C. Illustration of some pseudorandom phase sequences

For better understanding our scheme, the PPSs in the cases of modulating two and three classical fields is illustrated below. Using the method mentioned in section II, an m-sequence of length $4^2 - 1$ is generated by a primitive polynomial of the lowest degree over $GF(4)$, which is $\begin{bmatrix} 1 & 2 & 0 & 3 & 3 & 2 & 3 & 0 & 1 & 1 & 3 & 1 & 0 & 2 & 2 \end{bmatrix}$. Further we obtain a group that includes 16 PPSs of length 16: $\{\lambda^{(0)}, \dots, \lambda^{(15)}\}$, of which in exception to $\lambda^{(0)}$, all PPSs are independent and could be used to modulate classical fields to simulate quantum states of up to 15 particles. We could choose any two PPSs from the group for the simulation of two particles quantum state, for example,

$$\begin{aligned}\lambda^{(a)} &= \begin{bmatrix} 1 & 2 & 0 & 3 & 3 & 2 & 3 & 0 & 1 & 1 & 3 & 1 & 0 & 2 & 2 & 0 \end{bmatrix} \times \pi/2, \\ \lambda^{(b)} &= \begin{bmatrix} 2 & 1 & 2 & 0 & 3 & 3 & 2 & 3 & 0 & 1 & 1 & 3 & 1 & 0 & 2 & 0 \end{bmatrix} \times \pi/2.\end{aligned}\tag{35}$$

And we could also choose any three PPSs from the group for simulation of three particles quantum state, for example,

$$\begin{aligned}\lambda^{(a)} &= \begin{bmatrix} 1 & 2 & 0 & 3 & 3 & 2 & 3 & 0 & 1 & 1 & 3 & 1 & 0 & 2 & 2 & 0 \end{bmatrix} \times \pi/2, \\ \lambda^{(b)} &= \begin{bmatrix} 2 & 1 & 2 & 0 & 3 & 3 & 2 & 3 & 0 & 1 & 1 & 3 & 1 & 0 & 2 & 0 \end{bmatrix} \times \pi/2, \\ \lambda^{(c)} &= \begin{bmatrix} 2 & 2 & 1 & 2 & 0 & 3 & 3 & 2 & 3 & 0 & 1 & 1 & 3 & 1 & 0 & 0 \end{bmatrix} \times \pi/2.\end{aligned}\tag{36}$$

Using classical fields modulated the PPSs given above, we can efficiently simulate any quantum state of two or three particles.

V. CONCLUSIONS

In this paper, we have discussed a new scheme to simulate quantum states by using classical fields modulated with pseudorandom phase sequences. We first demonstrated that n classical fields modulated with n different PPSs can constitute a 2^n -dimensional Hilbert space that contains tensor product structure, which is similar with quantum systems. Further, by performing quadrature demodulation scheme, we obtained the mode status matrix of the simulating classical fields, based on which we proposed a sequence permutation mechanism to reconstruct the simulated quantum states. Besides, classical simulation of several typical quantum states was discussed, including product state, Bell states, GHZ state and

W state. We generalized our simulation and discussed the efficiency of our simulation finally. We conclude that quantum states can be efficiently simulated by using classical fields modulated with pseudorandom phase sequences. The research on simulation of quantum states may be important, for it not only provides useful insights into fundamental features of quantum mechanics, but also yields new insights into quantum computation and quantum communication.

Supported by the National Natural Science Foundation of China under Grant No 60407003, and National Basic Research Program 973 of China under Grant 2007CB307003. The authors thank Professors LIU Xu and TONG Li-Min for helpful discussion.

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Fig. 1 The PPS encoding scheme for one input field, where PNG denotes the pseudorandom number generator and PM denotes the phase modulator.

Fig. 2 The PPS quadrature demodulation scheme for one input field.

Fig. 3 The PPS quadrature demodulation scheme for one field with two orthogonal modes, where the gray block denote the mode splitter.

Fig. 4 The PPS quadrature demodulation scheme for multiple input fields, where the M block is shown in Fig. 3.

Fig. 5 The mode status matrix related to the fields and PPSs, where M_i^j is the element of $M \left(\tilde{\alpha}_i^j, \tilde{\beta}_i^j \right)$ for the i th classical field and the reference PPS $\lambda^{(j)}$.